Chapter 11: Ordinary Differential Equations

Learning Objectives:
(1) Solve first-order linear differential equations and initial value problems.
(2) Explore analysis with applications to dilution models.

1 Ordinary Differential Equations

Definition 1.1. An ordinary differential equation (ODE) is an equation involving one or more derivatives of an unknown function $y(x)$ of 1-variable. A differential equation for a multi-variable function is called a "partial differential equation" (PDE).

The order of an ordinary differential equation is the order of the highest derivative that it contains.

Example 1.1.

DIFFERENTIAL EQUATION		ORDER
$1\cdot\frac{dy}{dx} + 0\cdot y = 4x$	linear.	1
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y-1) = e^t$	linear	3
$1\cdot y' + 1\cdot y = 2x^2$	linear.	1

$1\cdot y^{(3)} + 0\cdot y^{(2)} + (-t)\,y' + (t)\,y = t + e^t$

↑ inhomogeneous term

Example 1.2. 1. $y\,y'' + e^y = x^2 \ln y'$ is a second order ODE. ← nonlinear

2. $f_2(x)y'' + f_1(x)y' + f_0(x)y = g(x)$, $f_2(x) \neq 0$. This is a second order *linear* ODE in the function $y(x)$. $g(x)$ is called the *inhomogeneous term*; the left hand side of the equation is called the *homogeneous part* of the this linear ODE; $f_2(x)y" + f_1(x)y' + f_0(x)y = 0$ is called the associated homogeneous linear ODE of the linear ODE given above. A linear ODE with inhomogeous term 0 is called a *homogeneous* linear ODE.

3. The ODE in 1. is non-linear. The second ODE in Example 1.1 is linear with inhomogeneous term e^t.

Remark. $\sum_{i=1}^{n} a_i x_i = b$, where a_i, b are constants ("coefficients") is said to be a linear equation in the variables $x_1, \ldots, x_n$. b is called the inhomogeneous term, and the equation is said to be homogeneous when $b = 0$. For differential equations, functions of x play the roles of "coefficients" $a_1, \ldots, a_n$, b, and $y^{(i)}$, $i = 0, 1, \ldots$ play the roles of "variables".

Definition 1.2. A function $y = y(x)$ is a **solution** of an ordinary differential equation on an open interval if the equation is satisfied identically on the interval when y and its derivatives are substituted into the equation.

Remark. The solution might not exist; it might not be unique.

Example 1.3. $y(x) = e^{2x}$ is a solution to the ODE $y'' - 4y' + 4y = 0$. $y(x) = 4e^{2x}$ is another solution.

Example 1.4. Find the solution of $\frac{d}{dx} y = 4x$, or equivalently, $y'(x) = 4x$.

Solution. Integrate both sides: $y(x) = \displaystyle\int 4x \, dx = 2x^2 + C$, where C is an arbitrary constant.

Then, $y = 2x^2 + C, \quad C \in \mathbb{R}$ is called general solution of $y'(x) = 4x$.

Choose any C, e.g. $C = 5$, we get a particular solution $y = 2x^2 + 5$. ■

For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function $y(x)$ at an arbitrary x-value x_0, say $y(x_0) = y_0$. This is called an initial condition, and the problem of solving a first-order equation subject to an initial condition is called a **first-order initial-value problem**.

Example 1.5.

$$\begin{cases} y'(x) = 4x \\ y(5) = 20 \end{cases}$$

is an initial value problem.

General solution $y = 2x^2 + C$ should satisfy the initial condition $y(5) = 20$, i.e.

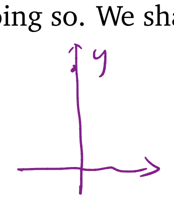
$$20 = 2(5)^2 + C \quad \Rightarrow \quad C = -30.$$

So, the unique solution to the initial value problem is $y = 2x^2 - 30$.

Remark. We saw that the general solution to a first order ODE typically involves an indeterminate constant C. More generally, the general solution to an n-th order ODE typically involves n indeterminate constants. An initial value problem for an n-th order ODE thus has n initial conditions, often of the form $y^k(x_0) = a_k$, $k = 0, 2, \ldots, n-1$, where x_0 and a_k are constants.

Solving a general ODE is typically very difficult, and there is no general algorithm for doing so. We shall discuss only some particularly simple cases.

Ex.



$y(t)$ position function of the ball : y': velocity ; y'': accelatation

$y'' = -g$ gravitional constant

initial conditions : $\begin{cases} y(0) = y_0 & \text{initial position} \\ y'(0) = v_0 & \text{initial velocity} \end{cases}$

2 Separation of Variables for first order ODEs

Definition 2.1 (Separable Equation).

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$\int h(y) \frac{dy}{dx} dx = \int g(x)\, dx$

$H(y) = \int h(y)\, dy = G(x) + C$

is called a separable equation.

For those separable differential equations, we can formally rewrite them in the form ("separation of variables"–each side involve one single variable)

$$\text{“}h(y)\, dy = g(x)\, dx\text{”} \tag{1}$$

Integrate both sides with respect to x and y respectively, we have

$$\int h(y)\, dy = \int g(x)\, dx \tag{2}$$

or, equivalently

$$H(y) = G(x) + C \tag{3}$$

where $H(x)$, $G(x)$ denote antiderivatives of $h(x)$ and $g(x)$ respectively, and C denotes a constant.

Example 2.1. Solve

$$(1) \quad \frac{dy}{dx} = \frac{2x}{y^2} \qquad\qquad (2) \quad \begin{cases} \dfrac{dy}{dx} = \dfrac{2x}{y^2}, \\ y(0) = 1. \end{cases}$$

Solution. (1) Separating variables and integrating yields

$$y^2 dy = 2x dx$$

$$\int y^2\, dy = \int 2x dx$$

or

$$\frac{1}{3}y^3 = x^2 + C$$

or, equivalently

$$y = \sqrt[3]{3(x^2 + C)}$$

plug in $x = 0, y = 1$

$\frac{1^3}{3} = 0^2 + C \Rightarrow C = \frac{1}{3}$

(2) The initial condition $y(0) = 1$ requires that $y = 1$ when $x = 0$. Substituting these values into our solution yields $C = \frac{1}{3}$ (verify). Thus, a solution to the initial-value problem is

$$y = \sqrt[3]{3x^2 + 1}.$$

■

Example 2.2. Solve

$$\frac{dy}{dx} = -4xy^3$$

Solution. (1) For $y \neq 0$, we can write the differential equation as

$$\frac{1}{y^3}\frac{dy}{dx} = -4x$$

Separating variables and integrating yields

$$\frac{1}{y^3}dy = -4xdx$$

$$\int \frac{1}{y^3}dy = \int -4xdx$$

or

$$-\frac{1}{2y^2} = -2x^2 + C$$

or, equivalently

$$y^2 = \frac{1}{4x^2 - 2C}$$

(2) Constant function $y = 0$ also satisfies the differential equation, since

$$0' = -4x \cdot (0)^3$$

Therefore, the solution is $y^2 = \frac{1}{4x^2-2C}$ or $y = 0$.

■

Remark. For $y' = g(x)h(y)$, divide both sides by $h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$.

Do not miss the particular constant solution $y = a$ that makes $h(a) = 0$.

Example 2.3. Solve $y' = 3x^2y$.

Solution. (1) For $y \neq 0$, it can be written as

$$\frac{dy}{y} = 3x^2\,dx$$

so

$$\begin{aligned}\int \frac{dy}{y} &= \int 3x^2\,dx \\ \ln|y| &= x^3 + C_1 \\ |y| &= e^{x^3}\cdot e^{C_1}, \quad C_1 \in \mathbb{R} \\ y &= \pm e^{x^3}\cdot e^{C_1}, \quad C_1 \in \mathbb{R} \\ y &= C_2 e^{x^3}, \quad C_2 \neq 0\end{aligned}$$

(2) Check: $y = 0$ is also a solution.

Therefore, the general solution is

$$y = Ce^{x^3}, \quad C \in \mathbb{R}$$

■

Example 2.4. Find a curve $y = y(x)$ on the $x - y$ plane that passes through $(0, 2)$ and whose tangent line at a point (x, y) has slope $2x^3/y^2$.

$y(0) = 2$

Solution. Since the slope of the tangent line is dy/dx, we have

$$\frac{dy}{dx} = \frac{2x^3}{y^2}$$

which is separable and can be written as

$$y^2 dy = 2x^3 dx$$

so

$$\int y^2 dy = \int 2x^3 dx \quad \text{or } \frac{1}{3}y^3 = \frac{1}{2}x^4 + C$$

$\frac{1}{3}2^3 = \frac{1}{2}\cdot 0^4 + C$

It follows from the initial condition that $y = 2$ if $x = 0$. Substituting these values into the last equation yields $C = \frac{8}{3}$ (verify), so the equation of the desired curve is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + \frac{8}{3}.$$

■

3 First-Order Linear Differential Equations

Recall: A 1st order linear ODE has the general form $a(x)y' + b(x)y = c(x)$, where $a(x) \neq 0$. We can always divide the whole equation by $a(x)$ and consider equivalently the equation $y' + \frac{b}{a}y = \frac{c}{a}$ wherever $a(x) \neq 0$. So we may restrict to equations of the form

$$\frac{dy}{dx} + p(x)\,y = q(x). \tag{4}$$

(1) If $q(x) = 0$ (homogeneous case),

$$\frac{dy}{dx} = -p(x)\,y.$$

$$\frac{dy}{dx} + p(x)y = 0, \qquad \text{separable equation!}$$

(2) For general $q(x)$, use integrating factors!

Idea: multiply the differential equation by a factor $\mu(x)$, then

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

Hope we can rewrite LHS in the form of $\frac{d}{dx}(\cdots)$, then the differential equation can be written as

$$\frac{d}{dx}(\cdots) = \mu(x)q(x) \qquad \text{separable equation!}$$

Check: $\boxed{\mu(x) = e^{\int p(x)\,dx}}$ works!

$$\frac{d\mu}{dx} = \mu p$$

$P(x)$ is an antiderivative of p

$\mu = e^{P}$

P_1 is another antiderivative of p

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu\frac{dy}{dx} + \frac{d\mu}{dx}y && \text{(product rule)}\\
&= \mu\frac{dy}{dx} + \mu p(x)y && \text{(chain rule)}\\
&= \mu q && \text{(apply equation)}
\end{aligned}$$

$P_1 = P + C$

$\mu_1 = e^{P_1} = e^{P} \cdot e^{C}$

$= e^{C}\mu$

So, $\mu y = \int \mu q\,dx$ and

$$\boxed{y = \frac{1}{\mu}\int \mu q\,dx}$$

$$= \frac{1}{\mu_1}\int \mu_1 q\,dx = \frac{1}{e^{C}\mu}\int (e^{C}\mu q)\,dx$$

Remark. There are infinitely many choices for $\mu(x) = e^{\int p(x)\,dx}$ (it involves an indefinite integral). Just pick any one!

The Method of Integrating Factors

Step 1. Calculate the integrating factor

$$\mu = e^{\int p(x)dx}.$$

Since any μ will suffice, we can take the constant of integration to be zero in this step.

Step 2. Multiply both sides of (4) by μ and express the result as

$$\frac{d}{dx}(\mu y) = \mu q(x).$$

Step 3. Integrate both sides of the equation obtained in Step 2 and then solve for y. Be sure to include a constant of integration in this step.

Example 3.1. Solve the differential equation

$$\frac{dy}{dx} - y = e^{3x}.$$

$y' + py = q$

$p = -1, \quad q = e^{3x}$

Solution. We have a first-order linear equation with $p(x) = -1$ and $q(x) = e^{3x}$.

$$\mu = e^{\int p(x)dx} = e^{\int(-1)dx} = e^{-x}.$$

Next we multiply both sides of the given equation by μ to obtain

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x}$$

which we can rewrite as

$$\frac{d}{dx}[e^{-x}y] = e^{2x}.$$

So

$$e^{-x}y = \frac{1}{2}e^{2x} + C$$

Finally, solving for y yields the general solution

$$y = \frac{1}{2}e^{3x} + Ce^{x}.$$

■

Exercise 3.1. Solve $y' + \overset{p}{2x}y = \overset{q}{4x}$.
Ans: $y = 2 + Ce^{-x^2}$.

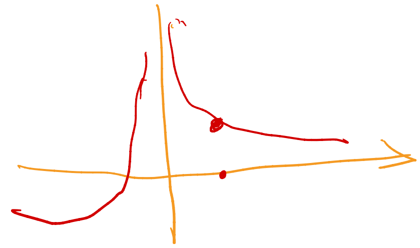
$\mu = e^{\int 2x\,dx} = e^{x^2}$

$\frac{d}{dx}(\mu y) = \mu y' + 2x\mu y = e^{x^2}4x$

$\mu y = \int 4x e^{x^2} dx = \int 2e^{u} du$

let $u = x^2$.
$du = 2x\,dx$

$= 2e^{u} + C = 2e^{x^2} + C \Rightarrow y = 2 + C \cdot e^{-x^2}$



Example 3.2. Solve the initial-value problem

$$x\frac{dy}{dx} - y = x, \quad y(1) = 2.$$

Solution. By dividing both sides by x to put the ODE in the standard form $y' + py = q$, we have

$$\frac{dy}{dx} \overset{p}{-\frac{1}{x}} y = \overset{q}{1} \quad \text{when } x \neq 0. \tag{5}$$

We shall look for solutions y with domain $\mathbb{R}\backslash\{0\} = (-\infty, 0) \cup \underline{(0, \infty)}$.

$\uparrow$ $x=1$ initial condition

In this problem, $p = -1/x$; so

$$\mu = e^{\int p(x)\,dx} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{|x|}.$$

Multiplying both sides of Equation (5) by this integrating factor yields

When $x > 0$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$$

or

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{1}{x}$$

Therefore

$$\frac{1}{x}y = \int \frac{1}{x}\,dx = \ln|x| + C$$

from which it follows that

$$y = x\ln|x| + Cx. \tag{6}$$

plug in $x=1$, $y=2$, $2 = 1 \cdot \ln 1 + C1 \Rightarrow C = 2$

By $y(1) = 2$, we have $C = 2$ (verify) *on the interval* $(0, +\infty) \ni 1$. So the general solution of the initial-value problem is

$$y = \begin{cases} x\ln x + 2x & \text{when } x > 0; \\ x\ln(-x) + Cx & \text{when } x < 0 \end{cases}$$

for an arbitray constant C. ■

Exercise 3.2. Solve the initial-value problem

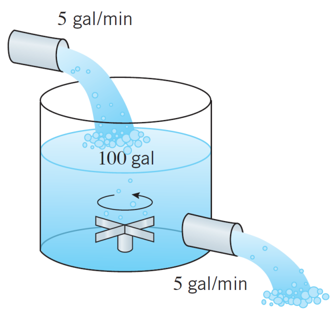
$$x\frac{dy}{dx} - y = x, \quad y(-1) = 2, \quad y(1) = 2.$$

has a unique solution.

4 Modeling with ODE

Example 4.1 (Mixing Problem)**.** At time $t = 0$, a tank contains 4 lb of salt dissolved in 100 gal of water. Suppose that brine containing 2 lb/gallon of salt is pumped into the tank at a rate of 5 gal/min. At the same time, that the well-mixed solution is drained from the tank at ~~the same rate~~. Find the amount of salt in the tank after 10 minutes.

4 gal/min



Solution.

$$
\begin{array}{llcl}
\text{Let} & y(t) & = & \text{amount of salt (lb) at time } t. \\
 & y(0) & = & 4 \text{ lb.} \\
\text{Aim:} & y(10) & = & ?
\end{array}
$$

Key: How $y(t)$ changes? or, $\dfrac{dy}{dt} = ?$ lb/min.

We always have

$$\boxed{\frac{dy}{dt} = \text{rate in} - \text{rate out.}}$$

where rate in is the rate at which salt enters the tank and rate out is the rate at which salt leaves the tank.

By the formula: $\boxed{\text{mass} = \text{volume} \times \text{concentration}}$, we have

$V(t)$ = volume of the solution in the tank
$= 100 + 5t - 4t$
$= 100 + t$

$$\text{rate in} = (2 \text{ lb/gal}) \cdot (5 \text{ gal/min}) = 10 \text{ lb/min.}$$

$$\text{rate out} = \left(\frac{y(t)}{100} \text{ lb/gal}\right) \cdot (5 \text{ gal/min}) = \frac{y(t)}{20} \text{ lb/min.}$$

$\left(\dfrac{y(t)}{V(t)}\right) \quad 4 \quad = \dfrac{4y}{100+t}$

Therefore, we have an initial first order linear ordinary differential equation

$$\begin{cases} \dfrac{dy}{dt} = 10 - \dfrac{y}{20} \\ y(0) = 4. \end{cases} \quad \text{or} \quad \frac{dy}{dt} + \frac{y}{20} = 10$$

$P = \frac{1}{20}$ $\quad q = 10$

The integrating factor for the differential equation is

$$\mu = e^{\int (1/20)dt} = e^{t/20}.$$

If we multiply the differential equation through by μ, then we obtain

$$\frac{d}{dt}(e^{t/20}y) = 10e^{t/20}$$

$$e^{t/20}y = \int 10e^{t/20}dt = 200e^{t/20} + C$$

$$y(t) = 200 + Ce^{-t/20}.$$

$4 = 200 + Ce^{0}$

Substituting $t = 0$ and $y = 4$ into $y(t)$ and solving for C yields $C = -196$, so

$$y(t) = 200 - 196e^{-t/20}.$$

At time $t = 10$, the amount of salt in the tank is

$$y(10) = 200 - 196e^{-10/20} \approx 81.1 \text{ lb}.$$

■

Remark. After sufficiently long time, as $t \to +\infty$, $y(t) \to 200$ lb.

Example 4.2. Modelling a pandemic: (SIR model)

`https://www.youtube.com/watch?feature=share&v=Qrp40ck3WpI&app=desktop`

Note: the number of infected grows exponentially in the initial stages (no intervention).

Coronavirus Cases Live Updates:

`https://www.youtube.com/watch?feature=share&v=Qrp40ck3WpI&app=desktop`